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LETTER TO THE EDITOR

Some special relations between 6-*j* and 9-*j* symbols

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Abstract. Attention is drawn to some special relations between 6-*j* and 9-*j* symbols. It is shown that one of the relations derived finds its application in the nuclear physics field.

Let us start with the well known sum rule introduced by Biedenharn (1953) and Elliott (1953):

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ l_1 & l_2 & l_3 \end{Bmatrix}\begin{Bmatrix} l_3 & j_1 & l_2 \\ l'_1 & l'_2 & l'_3 \end{Bmatrix} = \sum_x (-1)^{S+x}(2x+1)\begin{Bmatrix} l_1 & j_2 & l_3 \\ l'_3 & l'_2 & x \end{Bmatrix}\begin{Bmatrix} j_2 & j_3 & j_1 \\ l'_1 & l'_3 & x \end{Bmatrix}\begin{Bmatrix} l_1 & j_3 & l_2 \\ l'_1 & l'_2 & x \end{Bmatrix} \tag{1}$$

where $S = j_1+j_2+j_3+l_1+l_2+l_3+l'_1+l'_2+l'_3$. Introducing the parametrization $j_1 = l_3 = k_1-k$, $j_2 = l'_1 = l'_2 = k$, $j_3 = l_1 = k_1$, $l_2 = k_2$, $l'_3 = k_1-2k$, (1) can be written as:

$$\begin{Bmatrix} k_1-k & k & k_1 \\ k_1 & k_2 & k_1-k \end{Bmatrix}\begin{Bmatrix} k_1-k & k_1-k & k_2 \\ k & k & k_1-2k \end{Bmatrix}$$

$$= \sum_x (-1)^{k_1+k_2-k+x}(2x+1)\begin{Bmatrix} k_1 & k & k_1-k \\ k_1-2k & k & x \end{Bmatrix}^2\begin{Bmatrix} k_1 & k_1 & k_2 \\ k & k & x \end{Bmatrix}. \tag{2}$$

Here k_1, k_2 and k are non-negative integral or/and half-integral quantities, with the supplementary condition that none of the arguments, occurring in the 6-*j* symbols, may become negative. Considering the triangle relations between the arguments in the 6-*j* symbols, one observes that the summation index x can only take the value k_1-k. Using Racah's formula for *W*-coefficients (Racah 1942), modified for 6-*j* symbols, one obtains an analytic expression for the squared 6-*j* symbol in (2):

$$\begin{Bmatrix} k_1 & k & k_1-k \\ k_1-2k & k & k_1-k \end{Bmatrix}^2 = \frac{1}{(2k_1-2k+1)^2}. \tag{3}$$

Inserting (3) into (2) the following relation is obtained:

$$\begin{Bmatrix} k_1 & k_1 & k_2 \\ k & k & k_1-k \end{Bmatrix}$$

$$= (-1)^{2k_1-2k+k_2}(2k_1-2k+1)\begin{Bmatrix} k_1 & k_1 & k_2 \\ k_1-k & k_1-k & k \end{Bmatrix}\begin{Bmatrix} k_1-k & k_1-k & k_2 \\ k & k & k_1-2k \end{Bmatrix}. \tag{4}$$

This relation is valid for each value $k_1 \geq 2k$. For $k = 2k$ it is an identity.

In the standard textbooks dealing with angular momentum algebra (Edmonds 1957, Rose 1957, de-Shalit and Talmi 1963) one finds only relations where a 9-*j* symbol is expressed in terms of one 6-j symbol times a factor, for those 9-j symbols which have one argument 0 or 1. Using expression (4) another relation of the same kind can be deduced. Inserting $j_{11}=j_{33}=k_1-k$, $j_{31}=j_{12}=j_{23}=k$, $j_{21}=j_{32}=k_1$, $j_{22}=x$ and $j_{13}=k_1-2k$ into expression (6.4.3) of Edmonds (1957), one finds:

$$\begin{Bmatrix} k_1-k & k & k_1-2k \\ k_1 & x & k \\ k & k_1 & k_1-k \end{Bmatrix}$$

$$=\sum_s (-1)^{2s}(2s+1)\begin{Bmatrix} k_1-k & k_1 & k \\ k_1 & k_1-k & s \end{Bmatrix}\begin{Bmatrix} k & x & k_1 \\ k_1 & s & k \end{Bmatrix}\begin{Bmatrix} k_1-2k & k & k_1-k \\ s & k_1-k & k \end{Bmatrix},$$

which, due to (4), can be written as:

$$\begin{Bmatrix} k_1-k & k & k_1-2k \\ k_1 & x & k \\ k & k_1 & k_1-k \end{Bmatrix}=\sum_s \frac{(-1)^{2k_1-2k+3s}}{2k_1-2k+1}(2s+1)\begin{Bmatrix} k_1 & k_1 & s \\ k & k & k_1-k \end{Bmatrix}\begin{Bmatrix} k_1 & k_1 & s \\ k & k & x \end{Bmatrix}. \quad (5)$$

Here again k_1, k and x are non-negative integral or/and half-integral quantities. It is immediately clear that s can only take integral values. Making use of the analogue for 6-j symbols of the well known relation involving Racah's W-coefficients (Racah 1942), (5) reduces to:

$$\begin{Bmatrix} k_1-k & k & k_1-2k \\ k_1 & x & k \\ k & k_1 & k_1-k \end{Bmatrix}=\frac{(-1)^{k_1-k-x}}{2k_1-2k+1}\begin{Bmatrix} k_1 & k & k_1-k \\ k_1 & k & x \end{Bmatrix}. \quad (6)$$

This relation is again valid for each $k_1 \geqslant 2k$. At the same time it is an identity for $k_1=2k$.

Relations (4) and (6) can be helpful in analytic work. As an example we present the derivation of a formula used in nuclear physics, where relation (4) can be applied. Alaga and Paar (1976) mentioned, without proof, the following expression:

$$\langle NR=2N\|\alpha_2\|NR=2N\rangle = N(2R+1)\begin{Bmatrix} R & R & 2 \\ 2 & 2 & R-2 \end{Bmatrix}\langle 1\ 2\|\alpha_2\|1\ 2\rangle \quad (7)$$

for the reduced matrix elements of the vibrational amplitude α_2 between N-quadrupole phonon states of maximum angular momentum $R=2N$ in the anharmonic vibrator model.

Since an N-quadrupole phonon state can be developed in terms of a one-phonon and an $(N-1)$-phonon state:

$$|NR=2NM_R\rangle = \sum_{\mu' M_{R'}} \langle 2\mu'\ R-2\ M_{R'}|RM_R\rangle |1\ 2\ \mu'\rangle |N-1\ R-2\ M_{R'}\rangle,$$

the reduced matrix element can be written after some Racah algebra as follows:

$$\langle NR=2N\|\alpha_2\|NR=2N\rangle$$

$$=(2R+1)\begin{Bmatrix} R & R & 2 \\ 2 & 2 & R-2 \end{Bmatrix}\langle 1\ 2\|\alpha_2\|1\ 2\rangle$$

$$+(2R+1)\begin{Bmatrix} R & R & 2 \\ R-2 & R-2 & 2 \end{Bmatrix}\langle N-1\ R-2\|\alpha_2\|N-1\ R-2\rangle. \quad (8)$$

Since the second reduced matrix element on the right-hand side again occurs between states of maximum angular momentum $R-2=2(N-1)$, relation (8) can be re-applied $(N-2)$ times:

$$\begin{aligned}
&\langle NR=2N\|\alpha_2\|NR=2N\rangle \\
&\quad=(2R+1)\begin{Bmatrix} R & R & 2 \\ 2 & 2 & R-2 \end{Bmatrix}\langle 1\ 2\|\alpha_2\|1\ 2\rangle+(2R+1)(2R-3) \\
&\quad\times\begin{Bmatrix} R & R & 2 \\ R-2 & R-2 & 2 \end{Bmatrix}\begin{Bmatrix} R-2 & R-2 & 2 \\ 2 & 2 & R-4 \end{Bmatrix}\langle 1\ 2\|\alpha_2\|1\ 2\rangle \\
&\quad+(2R+1)(2R-3)(2R-7)\begin{Bmatrix} R & R & 2 \\ R-2 & R-2 & 2 \end{Bmatrix}\begin{Bmatrix} R-2 & R-2 & 2 \\ R-4 & R-4 & 2 \end{Bmatrix} \\
&\quad\times\begin{Bmatrix} R-4 & R-4 & 2 \\ 2 & 2 & R-6 \end{Bmatrix}\langle 1\ 2\|\alpha_2\|1\ 2\rangle+\ldots \\
&\quad+(2R+1)(2R-3)\ldots 13\begin{Bmatrix} R & R & 2 \\ R-2 & R-2 & 2 \end{Bmatrix} \\
&\quad\times\begin{Bmatrix} R-2 & R-2 & 2 \\ R-4 & R-4 & 2 \end{Bmatrix}\ldots\begin{Bmatrix} 6 & 6 & 2 \\ 2 & 2 & 4 \end{Bmatrix}\langle 1\ 2\|\alpha_2\|1\ 2\rangle \\
&\quad+2(2R+1)(2R-3)\ldots 13.9\begin{Bmatrix} R & R & 2 \\ R-2 & R-2 & 2 \end{Bmatrix} \\
&\quad\times\begin{Bmatrix} R-2 & R-2 & 2 \\ R-4 & R-4 & 2 \end{Bmatrix}\ldots\begin{Bmatrix} 6 & 6 & 2 \\ 4 & 4 & 2 \end{Bmatrix}\begin{Bmatrix} 4 & 4 & 2 \\ 2 & 2 & 2 \end{Bmatrix}\langle 1\ 2\|\alpha_2\|1\ 2\rangle. \qquad (9)
\end{aligned}$$

Relation (4) can be used to reduce each term appearing on the right-hand side of (9). For example the third term reads, after applying (4) to the last two 6-j symbols:

$$(2R+1)(2R-3)\begin{Bmatrix} R & R & 2 \\ R-2 & R-2 & 2 \end{Bmatrix}\begin{Bmatrix} R-2 & R-2 & 2 \\ 2 & 2 & R-4 \end{Bmatrix}\langle 1\ 2\|\alpha_2\|1\ 2\rangle$$

which is identical to the second term and to which (4) can be applied once more to give the result:

$$(2R+1)\begin{Bmatrix} R & R & 2 \\ 2 & 2 & R-2 \end{Bmatrix}\langle 1\ 2\|\alpha_2\|1\ 2\rangle.$$

So it follows that there are N terms on the right-hand side of (9), which are identical to the first term.

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